THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010G/H University Mathematics 2014-2015
Assignment 5

- Due date: 16 Apr, 2015 (before 17:00)
- Remember to write down your name and student number
- Please work on ALL questions below.

1. Given that $I_n = \int_0^1 (1-x^3)^n \, dx$, where n is a nonnegative integer. Show that for $n \geq 1$,
$$(3n+1)I_n = 3nI_{n-1}.$$

2. Evaluate
$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2n\pi}{n}\right).$$

3. By considering the substitution $t = \tan\frac{x}{2}$, find
$$\int \frac{1}{5+4\sin x}\, dx.$$

4. (a) Prove that
$$\int_0^1 \frac{u^4(1-u)^4}{1+u^2}\, du = \frac{22}{7} - \pi.$$
   (b) By considering the fact that $\frac{1}{2} \leq \frac{1}{1+u^2} \leq 1$ for any $0 \leq u \leq 1$, show that
$$\frac{22}{7} - \frac{1}{630} \leq \pi \leq \frac{22}{7} - \frac{1}{1260}.$$

5. Define $f(x) = (1-x)^n e^x$, where n is a positive integer.
   (a) Prove that $f(x)$ is decreasing for $0 \leq x \leq 1$.
   (b) Show that $\frac{1}{n!}\int_0^1 f(x)\, dx = e - \sum_{r=0}^{n} \frac{1}{r!}$.
   (c) Show that $0 < \int_0^1 f(x)\, dx < 1$. Hence, show that
$$\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{r!} = e.$$